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Derivation of integrable nonlinear evolution equations from the higher order NLS equation

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Abstract

A multiple scales method is used to derive the third-, fifth- and seventh-order KdV equations as amplitude equations from the integrable higher order nonlinear Schrödinger equation.

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1. Introduction

It is well known that a multiple scales analysis of the KdV equation (and, indeed a wide variety of equations) leads to the NLS equation for the modulated amplitude. In [6] Zakharov and Kuznetsov showed a much deeper correspondence between these integrable equations not only at the level of the equation, but also at the level of the linear spectral problem by showing that a multiple scales analysis of the Schrödinger spectral problem leads to the Zakharov–Shabat problem for the NLS equation. In [5], Dağ and Özer also showed the same relationship between the NLS equation and integrable fifth-order nonlinear evolution equations. In [4], Özer derived the recursion operator for the NLS equation from that of the KdV equation. In [15], Döken applied a multiple scales method to the NLS equation with the derivation of the KdV flow equations and their Hamiltonian densities.

In this paper, we apply a multiple scales method to derive the third-, fifth- and seventh-order KdV equations from the higher order NLS (HONLS) equation.

In section 2, we present some background material on the higher order NLS equation and integrable nonlinear evolution equations. In section 3 we first give a multiple scales method. Then we apply the method to the HONLS equation with the derivation of integrable nonlinear evolution equations.

Throughout the paper we make extensive use of Reduce [2] to calculate and simplify our results.

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2. Background material

In this section we present some background material on the HONLS and some integrable nonlinear evolution equations.

2.1. The HONLS equation

The NLS equation, which describes the time evolution of a slowly varying envelope, is encountered in various branches of physics and is known to be fundamental. In deriving the NLS equation as the envelope equation, we neglect higher order terms under appropriate physical assumptions. However, due to recent developments in optical technology, higher order corrections to the NLS equation have become necessary and important. Kodama and Hasegawa [7, 8] proposed the HONLS equation,

$$\mathrm{i}q_\tau = \mathrm{i}(a_0 q_{\xi\xi\xi} + a_3 q q^* q_\xi + a_4 q^2 q_\xi^*) - (a_1 q_{\xi\xi} + a_2 q^2 q^*) \tag{1}$$

which describes ultra-short pulse propagation in optical fibres, including higher order effects such as higher order dispersion. Here the star denotes the complex conjugation and parameters a_0, a_1, a_2, a_3, a_4 are real and satisfy the condition, $a_3^2 + a_4^2 \neq 0, s \neq 0$ if $a_0 = 0$. There is a known integrable case of the HONLS equation (1) which is solvable via the inverse scattering method: $a_0 = 0, a_1 \neq 0, a_3 \neq 0, a_4 = \frac{a_3}{2}$ (Kaup–Newell equation).

The Painlevé analysis of the HONLS equation, which has been carried out by a number of authors [9–14], strongly indicates that equation (1) is integrable in the above case.

2.2. Nonlinear evolution equations

We consider a system of nonlinear evolution equations

$$u_t = K[u] \tag{2}$$

where $K[u]$ is a locally defined function of u and its spatial derivatives. The well-known example of such an equation is the KdV equation

$$u_t = u_{xxx} + 6uu_x \tag{3}$$

which describes the propagation of waves on the surface of a shallow channel.

Fifth-order nonlinear evolution equations are of the form

$$u_t = K[u] = u_{xxxxx} + Auu_{xxx} + Bu_x u_{xx} + Cu^2 u_x. \tag{4}$$

According to [1, 3], there are only three known integrable cases of this class of equations:

$$u_t = \left(u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3\right)_x \tag{5}$$

$$u_t = \left(u_{xxxx} + 5uu_{xx} + \tfrac{5}{3}u^3\right)_x \tag{6}$$

$$u_t = \left(u_{xxxx} + 10uu_{xx} + \tfrac{15}{2}u_x^2 + \tfrac{20}{3}u^3\right)_x. \tag{7}$$

These are known respectively as Lax's fifth-order KdV flow, the Sawada–Kotera equation and the Kaup–Kupershmidt equation.

We also consider seventh- and higher order integrable nonlinear evolution equations in the following form [16]:

$$u_t = \left(u_{xxxxxx} + 14uu_{xxxx} + 28u_x u_{xxx} + 21u_{xx}^2 + 70u^2 u_{xx} + 70uu_x^2 + 35u^4\right)_x \tag{8}$$

$$u_{t_n} = R^n[u]u_x \tag{9}$$

where R is the recursion operator and is given by

$$R = \partial^2 + 4u + 2u_x \partial^{-1}.$$

3. The multiple scales method

In this section, following Zakharov and Kuznetsov [6], we use a multiple scales method to derive the KdV equation (3) and fifth- and seventh-order nonlinear evolution equations (4) and (8) from the HONLS equation (1) with the above integrability condition, respectively.

We now consider one of the integrable cases of HONLS equations which is known as the Kaup–Newell equation

$$\mathrm{i}q_\tau = \mathrm{i}\left(a_3 q q^* q_\xi + \frac{a_3}{2} q^2 q^*_\xi\right) - (a_1 q_{\xi\xi} + a_2 q^2 q^*) \tag{10}$$

and seek a solution by separating the phase and amplitude in the form:

$$q(\xi,\tau) = \mathrm{e}^{\mathrm{i}\theta(\xi,\tau)}\sqrt{N(\xi,\tau)} \qquad q^*(\xi,\tau) = \mathrm{e}^{-\mathrm{i}\theta(\xi,\tau)}\sqrt{N(\xi,\tau)}. \tag{11}$$

Inserting this assumed solution into the Kaup–Newell equation (10) and grouping the real and imaginary parts, we respectively obtain the following differential equations:

$$\begin{aligned} N_\tau &= \frac{1}{4N^2}(6a_3 N_\xi N^3 - 8a_1 N_\xi N^2 V - 8a_1 V_\xi N^3) \\ V_\tau &= \frac{1}{4N^3}\left(2a_1 N_{\xi\xi\xi} N^2 - 4a_1 N_{\xi\xi} N_\xi N + 2a_1 N_\xi^3 + 2a_3 N_\xi N^3 V \right. \\ &\qquad \left. + 4a_2 N_\xi N^3 - 8a_1 V_\xi V N^3 + 2a_3 V_\xi N^4\right) \end{aligned} \tag{12}$$

where $\theta(\xi,\tau)_\xi = V(\xi,\tau)$. Then we assume the following series expansions for solutions:

$$N = 1 + \sum_{n=1}^{\infty} \varepsilon^{2n} N_n(x, t_1, t_2, \ldots, t_n) \qquad V = \sum_{n=1}^{\infty} \varepsilon^{2n} V_n(x, t_1, t_2, \ldots, t_n). \tag{13}$$

We also define ξ, τ slow variables with respect to scaling parameters $\varepsilon > 0$ respectively as follows:

$$x = \varepsilon(\xi + 2\tau) \qquad t_1 = \varepsilon^3\tau \qquad t_2 = \varepsilon^5\tau, \ldots, t_n = \varepsilon^{2n+1}\tau. \tag{14}$$

We now substitute series expansions (13) and (14) into system (12) and equate coefficients at the powers of ε to zero separately. Then we end up with an infinite set of equations for N_n in the powers of ε for each n. If we let $\varepsilon \to 0$ and let the terms at minimal powers of ε vanish, by considering the case $n \geqslant 1$ we obtain for the coefficients ε^3,

$$V_1 = \frac{1}{4a_1}(3a_3 - 4)N_1 \tag{15}$$

taking

$$a_2 = \frac{1}{8a_1}\left(-3a_3^2 + 16a_3 - 16\right).$$

We now use (15) in the series expansion for the coefficient ε^5, we find

$$N_{1t_1} = \frac{1}{4(a_3 - 2)}\left[2a_1^2 N_{1xxx} + \left(18a_3 - 3a_3^2 - 24\right) N_1 N_{1x}\right]. \tag{16}$$

3.1. Derivation of the evolution equations

If we choose in (16)

$$a_3 = 5 \qquad a_1 = 1$$

we obtain

$$N_{1t_1} = \tfrac{1}{6}\left(N_{1xxx} - \tfrac{9}{2} N_1 N_{1x}\right) \tag{17}$$

or making the following transformation

$$t_1 \to \tfrac{1}{6}t_1 \qquad N_1 \to -\tfrac{4}{3}u$$

we derive the KdV equation (3) with $t \to t_1$,

$$u_{t_1} = u_{xxx} + 6uu_x \tag{18}$$

where $u = u(x, t_1, t_2, \ldots, t_n)$. Inserting (17), taking $N_1 \to -\frac{4}{3}u$ into series expansion at the coefficient ε^7, we obtain

$$N_{1t_2} = \tfrac{1}{4752}\big[297N_{2t_1} + 24N_{1xxxxx} + 56N_1N_{1xxx} + 528N_{1x}N_{1xx} + 11\,682N_2N_{1x} + 1038N_1^2N_{1x} - 5544V_2N_{1x} - 216V_{2xxx} - 1296V_{2x}N_1 + 1188V_{2t_1}\big]. \tag{19}$$

If we take in equation (19) for N_2, V_2 as

$$N_2 = k_{11}N_{1xx} + k_{11}N_1^2 \qquad V_2 = k_{21}N_{1xx} + k_{22}N_1^2 \tag{20}$$

then choosing

$$k_{11} = -\tfrac{7}{6} \qquad k_{12} = \tfrac{95}{48} \qquad k_{21} = -\tfrac{79}{24} \qquad k_{22} = \tfrac{949}{192}$$

we get

$$N_{1t_2} = \tfrac{1}{216}\left(N_{1xxxxx} + 10N_1N_{1xxx} + 20N_{1x}N_{1xx} + 30N_1^2N_{1x}\right). \tag{21}$$

Here using the following transformation

$$t_2 \to \tfrac{1}{216}t_2 \tag{22}$$

we derive the fifth-order Lax's KdV equation as

$$u_{t_2} = u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x. \tag{23}$$

If we choose in equation (20)

$$k_{11} = -\tfrac{337}{216} \qquad k_{12} = \tfrac{755}{288} \qquad k_{21} = -\tfrac{3779}{864} \qquad k_{22} = \tfrac{7549}{1152}$$

and use transformation (22), then equation (19) forms Sawada–Kotera's equation

$$u_{t_2} = u_{xxxxx} + 5uu_{xxx} + 5u_xu_{xx} + 5u^2u_x \tag{24}$$

or choosing in equation (20)

$$k_{11} = -\tfrac{173}{144} \qquad k_{12} = \tfrac{145}{72} \qquad k_{21} = -\tfrac{1951}{576} \qquad k_{22} = \tfrac{1451}{288}$$

and using transformation (22), we obtain Kaup–Kupershmidt's equation

$$u_{t_2} = u_{xxxxx} + 10uu_{xxx} + 25u_xu_{xx} + 20u^2u_x \tag{25}$$

from equation (19).

Similarly taking N_3, V_3 with $N_1 \to -\frac{4}{3}u$ for the coefficient ε^9, as

$$N_3 = t_{11}u_{xxxx} + t_{12}uu_{xx} + t_{13}u_x^2 + t_{14}u^3$$
$$V_3 = t_{21}u_{xxxx} + t_{22}uu_{xx} + t_{23}u_x^2 + t_{24}u^3$$

where

$$t_{11} = \tfrac{717\,071}{349\,920} \qquad t_{12} = -\tfrac{116\,271}{12\,960}, \ldots$$

and making the transformation

$$t_3 \to \tfrac{1}{7776}t_3$$

we obtain the seventh-order Lax's KdV equation as

$$u_{t_3} = u_{xxxxxxx} + 14uu_{xxxxx} + 42u_xu_{xxxx} + 70u_{xx}u_{xxx} + 70u^2u_{xxx} + 280uu_xu_{xx} + 70u_x^3 + 140u^3u_x. \tag{26}$$

We, therefore, get the KdV equation (3) and fifth- and seventh-order integrable nonlinear evolution equations which are given by (5), (6), (7) and (8) from the HONLS equation (1) by using the multiple scales method. If we continue calculations as before, we can obtain higher order integrable nonlinear evolution equations (9).

We have found $t_1, t_2, t_3, \ldots$ derivatives of the amplitude $u(x, t_1, t_2, \ldots, t_n)$ at each level of perturbation. This raises an important question of consistency. To discuss this question we have to check the following compatibility condition:

$$\frac{\partial u}{\partial t_i \partial t_j} = \frac{\partial u}{\partial t_j \partial t_i} \qquad i, j = 1, 2, 3, \ldots, n \tag{27}$$

for equations (18), (23), (26), (24) and (25). In our case, this compatibility condition (27) is satisfied for each flow.

4. Conclusion

We have used a multiple scales method to provide a new derivation of the famous KdV equation and higher order integrable nonlinear evolution equations from the HONLS equation. Thus there exists a relation between the NLS type of Kaup–Newell equation (10) with the KdV equation (3) and the integrable cases of fifth-order (4) and seventh-order nonlinear (8) evolution equations. The details are given in [15].

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